Excercise 5

Concurrency Theory

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1 Lexicographic Order is a well-pseudo-order

Let $a = (a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{N}^6$ and $b = (b_1, b_2, b_3, b_4, b_5, b_6) \in \mathbb{N}^6$ be two states.
Their first difference is at: $i_{a,b} := \max\{x \in [1, 6] \mid \forall i < x : a_i = b_i\}$.
$a \preceq b \iff a_{i_{a,b}} \leq b_{i_{a,b}}$. Note that in case there is none at all, we have $i_{a_b} = 6$. Hence the last digit decides.

Theorem 1. *$\preceq$ is a well-pseudo-order.*

Proof. **Reflexive** Let $a \in \mathbb{N}^6$. It is $i_{a,a} = 6$ and $a_6 = a_6$. $\preceq$ is reflexive ✓

Transitive Let $a \preceq b$ and $b \preceq c$. Thus $a_{i_{a,b}} \leq b_{i_{a,b}}$ and $b_{i_{b,c}} \leq c_{i_{b,c}}$.

Case I ($i_{a,b} = i_{b,c}$):
Here $i_{a,c} = i_{a,b} = i_{b,c}$ and $a_{i_{a,c}} \leq c_{i_{a,c}}$. Hence $a \preceq c$.

Case II ($i_{a,b} < i_{b,c}$):
Here $a_{i_{a,b}} \leq b_{i_{a,b}} = c_{i_{a,b}}$. For all $j < i_{a,b}$ we have $a_j = b_j = c_j$. Thus $i_{a,c} = i_{a,b}$. So due to $a_{i_{a,c}} \leq c_{i_{a,c}}$ it is $a \preceq c$.

Case III ($i_{a,b} > i_{b,c}$):
Here $a_{i_{b,c}} = b_{i_{b,c}} \leq c_{i_{b,c}}$. For all $j < i_{b,c}$ we have $a_j = b_j = c_j$. Thus $i_{a,c} = i_{b,c}$. So due to $a_{i_{a,c}} \leq c_{i_{a,c}}$ it is $a \preceq c$.

Antisymetric Let $a \preceq b$ and $b \preceq a$. Let $i := i_{a,b} = i_{b,a}$. It is $a_i \leq b_i \leq a_i$. Thus $a_i = b_i$, which can only be the case at the very end, i.e. $i = |a| = |b| = 6$. So $a = b$.

Well First note that a and b are always comparable under $\preceq$. So $a \not\preceq b \implies a \succ b$.

Let $a, b \in \mathbb{N}^d$. We will show via induction over d that every decreasing sequence, with respect to $\preceq$, is finite.

Induction Basis $d = 1$
Here $a, b \in N$ and $\preceq = \leq$. Obviously there is no infinite sequence $a^1 > a^2 > \dots$ for $a^i \in \mathbb{N}$. ✓

Induction Hypothesis For one d we know $\preceq$ allows only finite decreasing sequences.

Induction Step $d+1 \curvearrowright d$

Observe any decreasing sequence $a^1 \succ a^2 \succ \dots$ with $a^i \in \mathbb{N}^{d+1}$. Note that for all a^i the first element has to be at least $a_1^1 \geq a_1^i$.

$a_1^1 = a_i^i$ cannot be true for all i since due to the induction hypotheses there are only finite decreasing sequences over $\mathbb{N}^d$.

Let us look at the first i where $a_1^1 > a_1^i$: Due to the induction hypotheses the sequence in between was finite.

Since we can decrease a_1^i only a finite amount of times the while sequence is finite. ✓

□

2 Game is finite

As we just have proven there is infinite decreasing sequence over $\mathbb{N}^6$ with repsect to $\preceq$. Let us look at the two operation. We will show that both decrease the state.

Move Let

$$m(a,i)_k = \begin{cases} a_k & i \neq k \neq i+1 \\ a_k - 1 & i = k \\ a_k + 2 & i+1 = k. \end{cases}$$

Observe any $b = m(a,i)$. For all $j > i$ it is $b_j = a_j$ and $a_i > b_i$. Hence $i_{a,b} = i$ and $a \succ b$.

Swap Let

$$s(a,i,j,k)_l = \begin{cases} a_i - 1 & l = i \\ a_k & l = j \\ a_j & l = k \\ a_l & \text{otherwise} \end{cases}$$

Again observe any $b = s(a,i,j,k)$. Again for all $j > i$ it is $b_j = a_j$ and $a_i > b_i$. Hence again $i_{a,b} = i$ and $a \succ b$.

Thus every sequence of moves is finite.

3 Compatibility

a) Lexicographic

Let $a = (1,1,1,5,3,1)$, $b = s(a,3,4,5) = (1,1,0,3,5,1)$ and $a \preceq a' = (1,1,1,6,2,0)$. Observe $b' := s(a',3,4,5) = (1,1,0,6,2,0)$. $a' \not\preceq b'$. Thus we found counterexample.

b) Component Wise

Let a and $a' \in \mathbb{N}^6$ s.t. $a \leq a'$.
Let's first look at the move operation: Let $b = m(a, i)$ and $b' = m(b, i)$.

$$\begin{aligned} b_i = a_i - 1 \leq a'_i - 1 = b'_i \\ b_{i+1} = a_i + 2 \leq a'_i + 2 = b'_i \\ \forall j \, \{i, i+1\} : b_j = a_j \leq a'_j = b'_j \end{aligned}$$

Hence $b \leq b'$.
Now let's first look at the swap operation: Let $b = m(a, i, j, k)$ and $b' = s(b, i, j, k)$.

$$\begin{aligned} b_i = a_i - 1 \leq a'_i - 1 = b'_i \\ b_j = a_k + 2 \leq a'_k + 2 = b'_j \\ b_k = a_j + 2 \leq a'_j + 2 = b'_k \\ \forall j \, \{i, j, k\} : b_j = a_j \leq a'_j = b'_j \end{aligned}$$

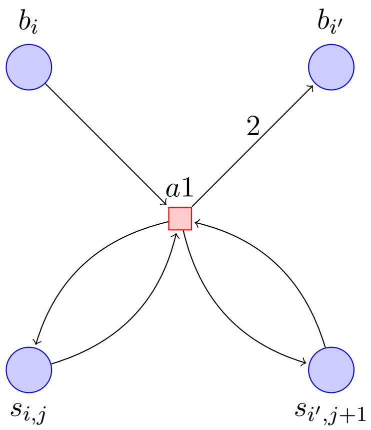
Hence $b \leq b'$.

4 Petri Net

Define $N = \{1, \ldots, n\}$.

For each bucket $i \in N$, we have one place b_i that stores the number of coins in this bucket, and n places $s_{i,j}$, $j \in N$, that stores the position of the bucket: Iff there is a token in place $s_{i,j}$, then bucket i is at position j. So, all in all, we have $n^2 + n$ places.

For the first action we add the following transitions:
For every $i \in N, i' \in N \setminus \{i\}, j \in N \setminus \{n\}$:



For the second action we add the following transitions:
For every $i \in N, i' \in N \setminus \{i\}, i' \in N \setminus \{i, i'\}, j < j' < j'' \in N$: