**Excercise 7**

# Concurrency Theory

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## 2 Bounded CSM vs. NFA

### a) NFA simulates a bounded CSM

Let the CSM be formed out of the machines $M_1$ to $M_N$. $\Sigma$ is the set of messages. The channels are bounded by $k$.
Let $W_k = \bigcup_{0 \leq i \leq k} \Sigma^i$ be the set of all words over $\Sigma$ up to a length $k$.
Let $R = (Q, A, \Delta, s, F)$ be the resulting NFA. In the states $Q$ we have to decode the state of all $N$ machines as well their $N$ channels. Thus

$$Q = \{(C_1, \dots, C_N, s_1, \dots, s_N) \mid C_i \in W_k \wedge s_i \in S_i\} = W_k^N \times S_1 \times \dots \times S_N.$$

The alphabet $A = N \times ((N \times \{!\} \times \Sigma) \cup (\{?\} \times \Sigma))$ decodes the sending of receiving messages. Additionally the very first component stores who sended/received.

Let us look at any transition $t = A \overset{\mathsf{M_j!a}}{\to} B \in \Delta_i$ in Machine $M_i$: We have to take every possible content of each channel into account. Then we have to ensure that states $s_i$ changes from $A$ to $B$ while all the other stay unchanged. But here we again have to take every possible state into account, i.e. $s_l$ might be any in $S_l$, for every $l$. Finally we have to add $\mathsf{a} \in \Sigma$ to the channel of $M_j$, namely $C'_j$. Since we have to ensure that $\mathsf{a}$ fits into the channel $C_j \in W_{k-1}$ this time. Formally we get those transitions:

$$\delta(i, A, B, \mathsf{M_j!a}) = \left\{ (C_1, \dots, C_N, s_1, \dots, s_N) \overset{\mathsf{i{:}j!a}}{\to} (C'_1, \dots, C'_N, s'_1, \dots s'_N) \;\middle|\; * \right\}$$

$$\begin{aligned} * \iff & \forall_{l \neq j} C_l = C'_l \in W_k \\ & \wedge \forall_{l \neq i} s_l = s'_l \in S_l \\ & \wedge C_j \in W_{k-1} \\ & \wedge C'_j = C_j \cdot \mathsf{a} \\ & \wedge s_i = A \\ & \wedge s'_i = B \end{aligned}$$

Similarly we have to look at receiving transitions $t = A \overset{?\mathsf{a}}{\to} B \in \Delta_i$ in Machine $M_i$: Here we have to remove a $\mathsf{a} \in \Sigma$ from $C_i$. Formally we get those transitions:

$$\delta(i, A, B, ?\mathsf{a}) = \left\{ (C_1, \dots, C_N, s_1, \dots, s_N) \overset{\mathsf{i:?a}}{\to} (C'_1, \dots, C'_N, s'_1, \dots s'_N) \;\middle|\; * \right\}$$

$$\begin{aligned} * \iff & \forall_{l \neq i} C_l = C'_l \in W_k \\ & \wedge \forall_{l \neq i} s_l = s'_l \in S_l \\ & \wedge C'_i \in W_{k-1} \\ & \wedge C_i = C'_i \cdot \mathsf{a} \\ & \wedge s_i = A \\ & \wedge s'_i = B \end{aligned}$$

Now we can use those $\delta$ to define $\Delta$, the transitions of $R$, as union over all of these:

$$\Delta = \bigcup_{0 \le i \le N} \bigcup_{A \overset{m}{\to} B \in \Delta_i} \delta(i, A, B, m).$$

If some $M_i$ can reach some state $s_i \in S_i$ this corresponds to at least one state in

$$F_{i,s_i} = \left\{ q \in Q \;\middle|\; q = (C_1, \dots, C_N, s'_1, \dots, s'_N) \wedge s'_i = s_i \right\}$$

being reachable.

$$\begin{aligned} |Q| &= |W_k|^N \cdot \prod |S_i| \\ &\le |W_k|^N \cdot m^N \\ &\le \left( \sum_{i \le k} |\Sigma|^i \right)^N \cdot m^N \\ &\le \left( \frac{|\Sigma|^{k+1} - 1}{|\Sigma| - 1} \cdot m \right)^N \\ &< \infty \end{aligned}$$

## b) Bounded CSM simulates a NFA

This direction is very simple. We only need one Machine $M$. The bound $k = 1$ is tight. The idea is that $M$ basically is able to do anything the NFA $R = (Q, \Sigma, \delta, s, F)$ can do. Any transition in $R$ will corresponds to 2 consecutive transitions in $M$. First he sends something to himself just to receive it immediately. $k$ being tight, i.e. 1, ensures that only one of $R$'s transitions can be simulated at a time.
$A \overset{\mathsf{a}}{\to} B$ becomes $A \overset{\mathsf{M!a}}{\to} A' \overset{?\mathsf{a}}{\to} B$.

The states of $M$ are $S = Q \cup Q'$, where $Q' = \{q' \mid q \in Q\}$ is a set of *marked* states. The transitions of $M$ are

$$\Delta = \left\{ A \overset{\mathsf{M!a}}{\to} A' \;\middle|\; t = A \overset{\mathsf{a}}{\to} B \in \delta \right\} \cup \left\{ A' \overset{\mathsf{?a}}{\to} B \;\middle|\; t = A \overset{\mathsf{a}}{\to} B \in \delta \right\}.$$

If $R$ can reach state $f \in Q$ using some transitions $t_1, t_2, \ldots, t_k$ (where $t_i = s_{i-1} \overset{a_i}{\to} s_i$) we can do $s_0 \overset{M!a_1}{\to} s_0' \overset{?a_1}{\to} s_1 \ldots s_{k-1}' \overset{?a_k}{\to} f$ analogously in $M$.
$|\Sigma|$ cannot be expressed in terms of $|Q|$ since the number of messages is exactly the alphabet size of $R$, which is unrelated to $|Q|$. There might be $n \in \mathbb{N}$ messages while there is only 1 state, or the other way round.
$N(Q) = k(Q) = 1$
$m(Q) = 2|Q|$.