Parameterised Systems

Daniel Stan, *Automated Reasoning group, TUK*

Parameterised System

Motivations: analysis of a whole family of systems $(\mathscr{S}_p)_p$.
$p \in Param$ is a parameter.

- **Synthesis**: "Does there exists $p \in Param$ such that $\mathscr{S}_p \models \varphi$?"
- **Validity**: "Does $\mathscr{S}_p \models \varphi$ for all p?"

Parameterised System

$\xrightarrow{t < c}$

Different types of parameters:

- Guard constants values (in timed automata): $Param \subseteq \mathbb{R}^+$;
- Probability values (in stochastic systems): $Param \subseteq [0,1]$; $\xrightarrow{p}$
- ...
- Number of interacting copies of processes: $Param = \mathbb{N}$.

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Today: **Validity** of parameterised systems composed with **arbitrary many copies** using **well-quasi order** techniques.

Arbitrary many copies

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Definition

A parameterised system $\mathscr{S} = (S, I, \Sigma, \rightarrow)$ is a LTS $(S, \Sigma, \rightarrow)$ equipped with

- A WQO $\preceq$ over S;
- An upward-closed set of initial states I;
- A norm function $|.| : S \rightarrow \mathbb{N}$ such that $\forall (s,t), s \leq t \Rightarrow |s| \leq |t|$

We write $S_k = \{s \in S \mid |s| = k\}$ and $\mathscr{S}_k$ the LTS restricted to initial states in $I \cap S_k$.

$S \geq \ell$

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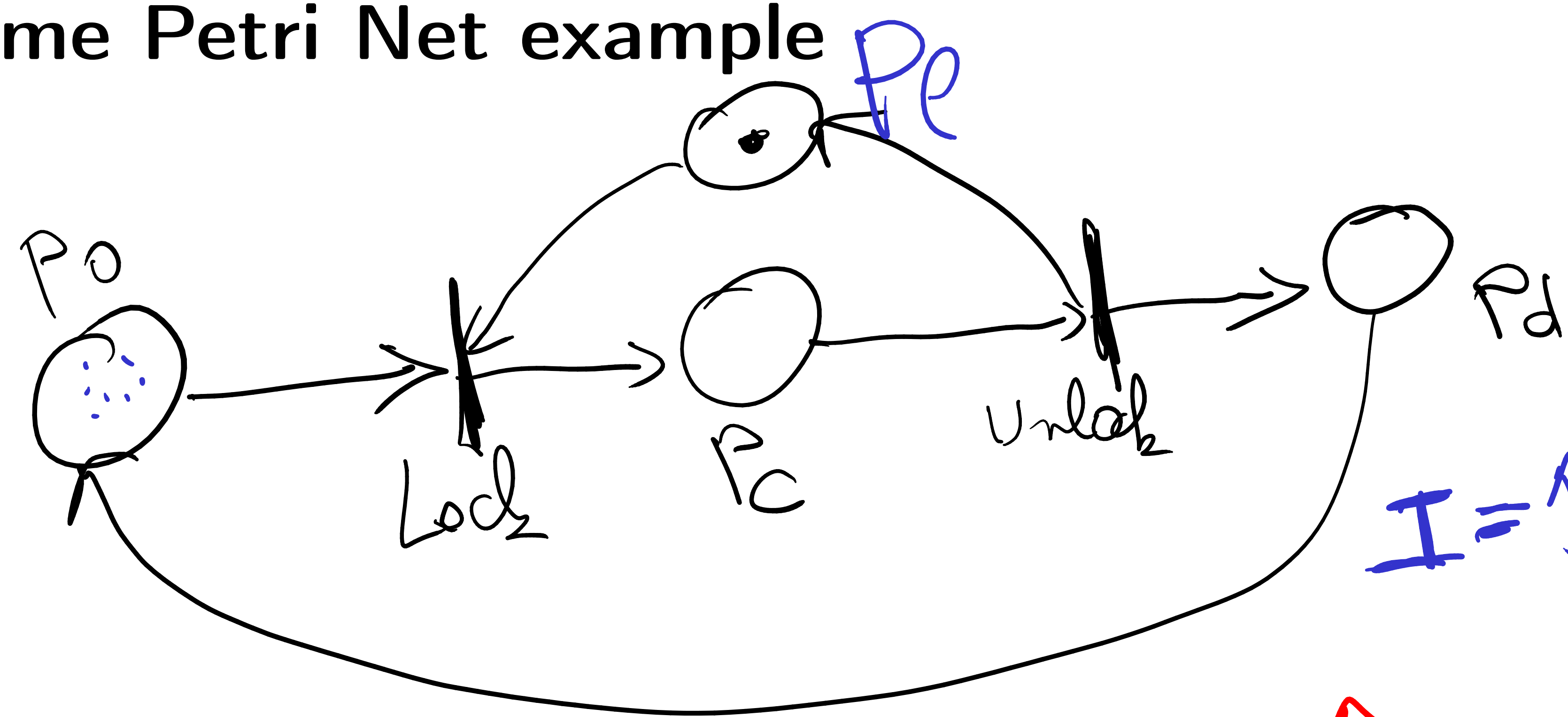
Not necessarily a WSTS!

Some Petri Net example



$I = \uparrow p_0 + p_l$

$Bad = \uparrow 2p_c$

$|m| = m(p_0) + m(p_c) + m(p_d)$

$I = \{k \cdot p_0 + p_l \mid k \in \mathbb{N}\}$,

$|M| = \ldots$,

$M \leq M'$ if $\forall p \in P, M(p) \leq M'(p)$

Some Petri Net example

$I = \{k \cdot p_0 \mid k \in \mathbb{N}\}$, ← upw closed

$|M| = \ldots$,

$M \preceq M'$ if $\forall p \in P, M(p) \leq M'(p)$ **and** $support(M) = \{p \mid M(p) > 0\} = \{p \mid M'(p) > 0\} = support(M')$.

(still a WQO)

Remark:

$|M|$ is not necessarily the number of tokens in M.

Parameterised Systems (Our definition today)

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- An upward-closed set of initial states I;
- A norm function $|.| : S \rightarrow \mathbb{N}$ such that $\forall (s,t), s \leq t \Rightarrow |s| \leq |t|$.
- $\rightarrow$ **preserves the norm**
- S_k **is finite for any** k**.**

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Cut-off property

Definition
A parameterised system $(\mathscr{S}_n)_n$ has a **cut-off property** for φ if there exists a **cut-off bound** $n \in \mathbb{N}$, such that

$$\forall k > n, \mathscr{S}_k \vDash \varphi \Leftrightarrow \mathscr{S}_n \vDash \varphi$$

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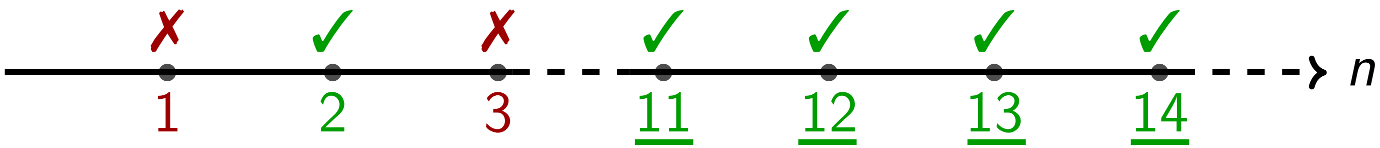

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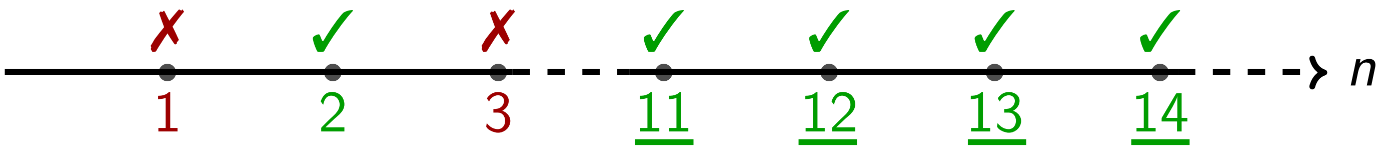

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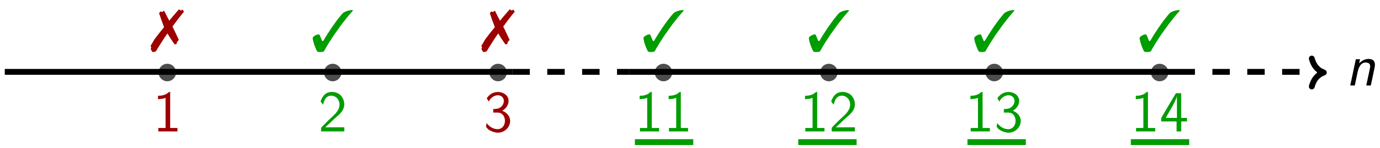

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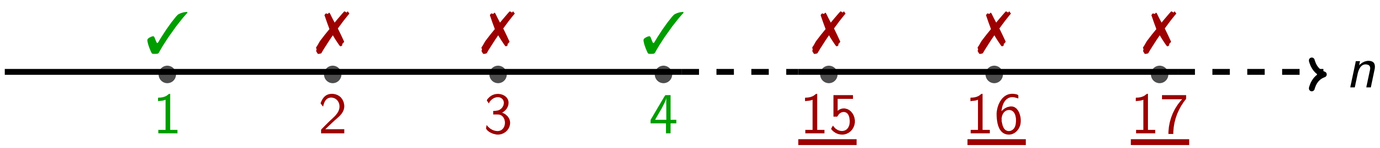

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- **Karp-Miller tree** from $M_0 = \omega \cdot p_0$.

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- **Karp-Miller tree** from $M_0 = \omega \cdot p_0$.
- (Recall week 3): for any M, if there is M' in $KarpMillerTree(N)$ with $M \leq M'$, then N can cover M.
- **Conclusion**: the problem is EXPSPACE-complete.

Recall: Backward coverability for WSTS

function BackCoverFormal($\boldsymbol{N}, \boldsymbol{p_0}, \boldsymbol{Bad}, \leq$)

$U \leftarrow Bad$

$U' \leftarrow \emptyset$

while $U \neq U'$ **do**

$U' \leftarrow U$

$U \leftarrow U \cup \mathrm{Pre}(U)$ // U stays up-closed

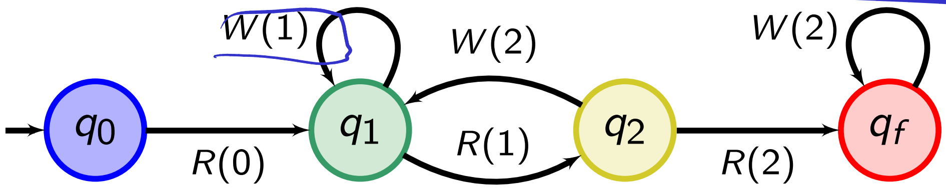
return $\exists k : k \cdot p_0 \in U$

Non-atomic operations on shared variables

Inspired by:

- *Model checking parameterized asynchronous shared-memory systems.*
  Antoine Durand-Gasselin, Javier Esparza, Pierre Ganty, Rupak Majumdar, 2017
- *Reachability in Networks of Register Protocols under Stochastic Schedulers*
  Patricia Bouyer, Nicolas Markey, Mickael Randour, Arnaud Sangnier, Daniel Stan, 2016

The model



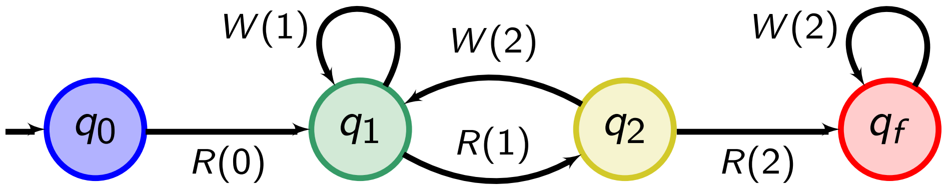
Register protocol with $D = \{0, 1, 2\}$.

Definition: register protocol

$\mathscr{P} = \langle Q, D, q_0, d_0, q_f, T \rangle$

- $\langle Q, q_0, q_f, T \rangle$ is a finite state automaton;
- D finite dataset for the shared register;
- d_0 an initial value;
- $T \subseteq Q \times \{R, W\} \times D \times Q$ set of transitions, labelled by read and write operations over D.

The model



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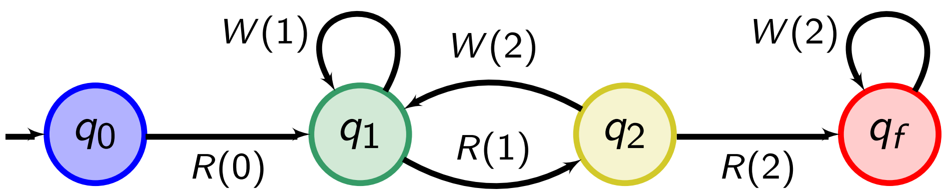
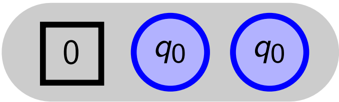


Network for two processes (self-loops omitted).



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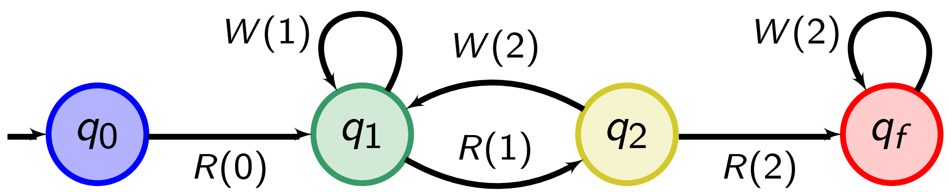

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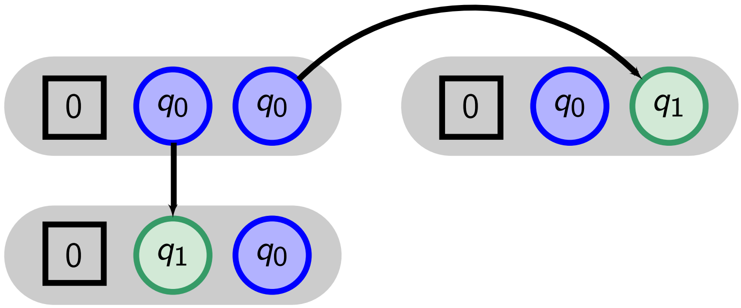

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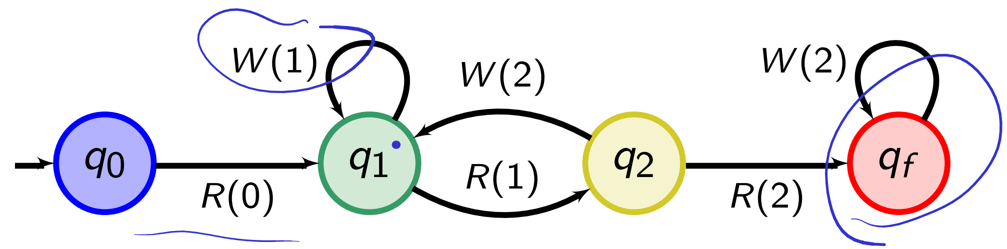

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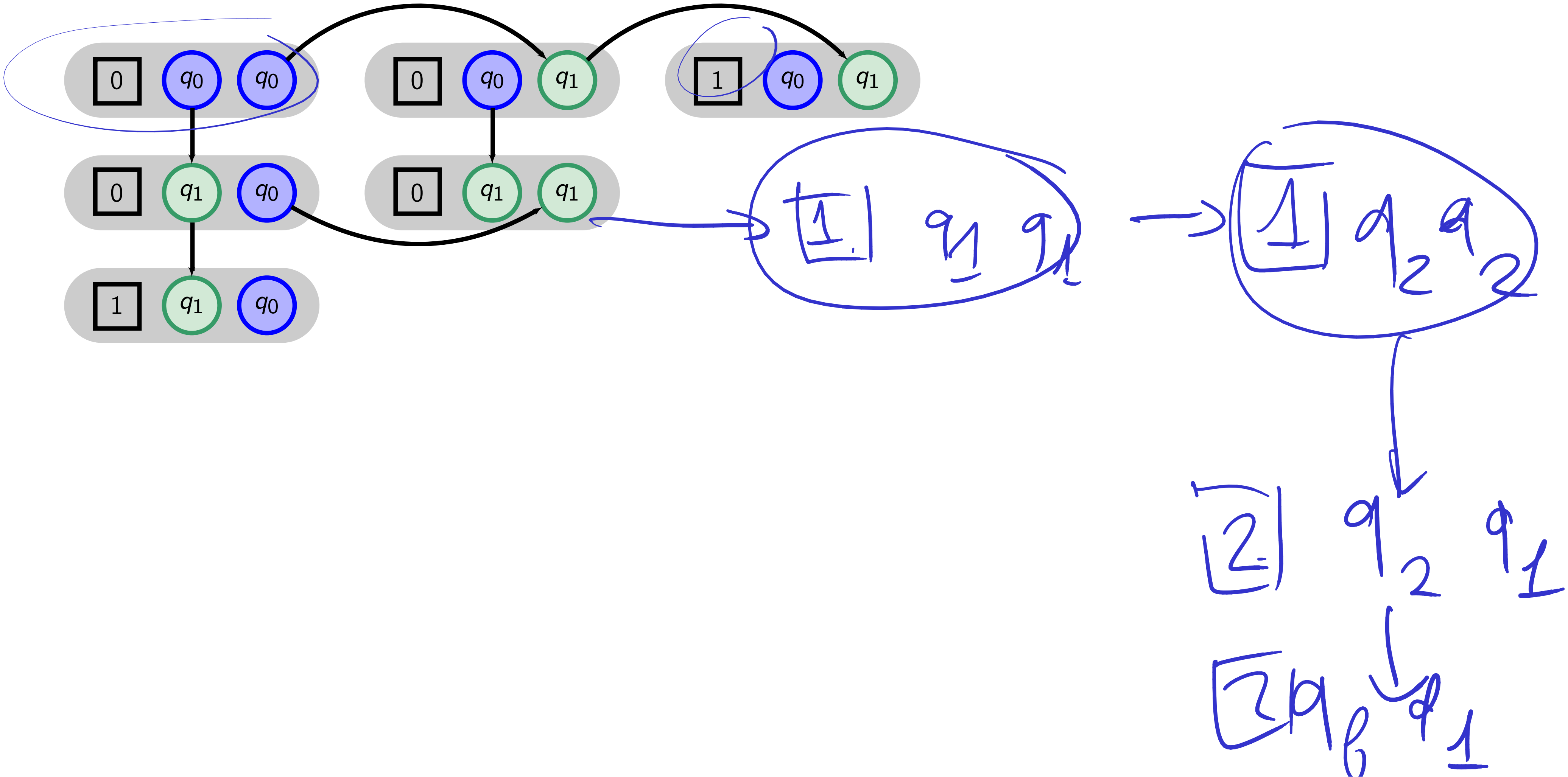

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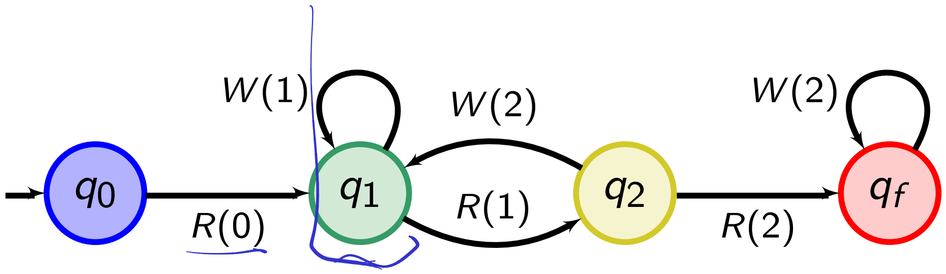

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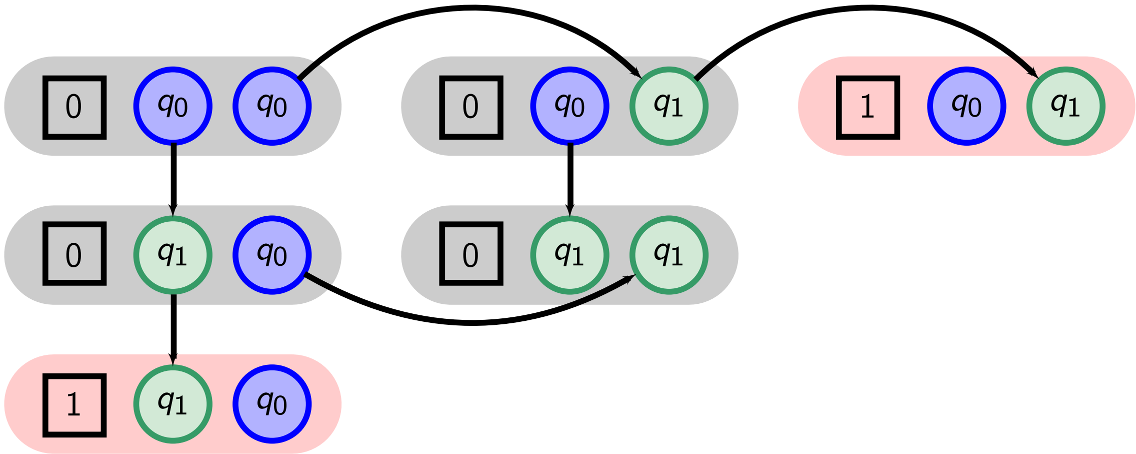

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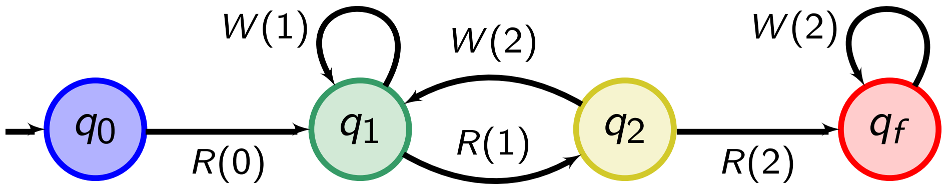

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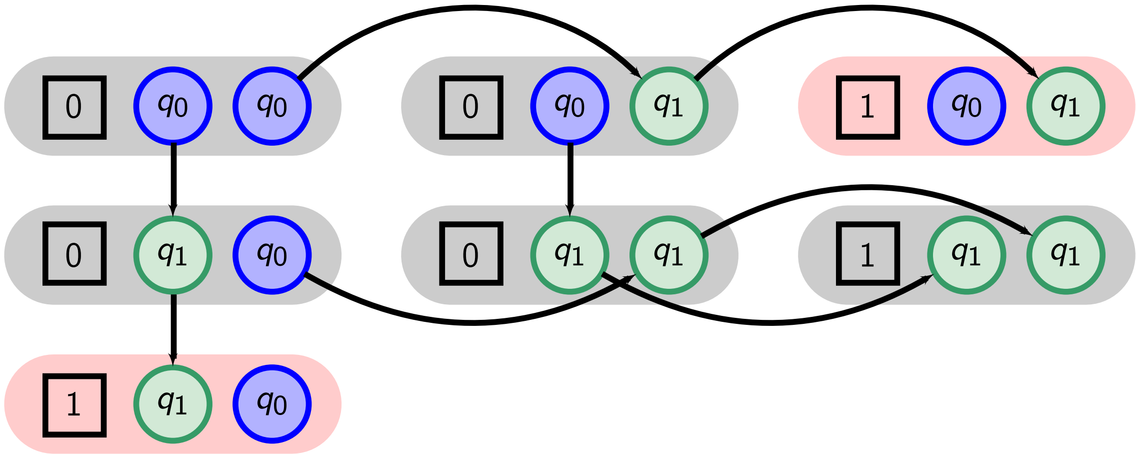

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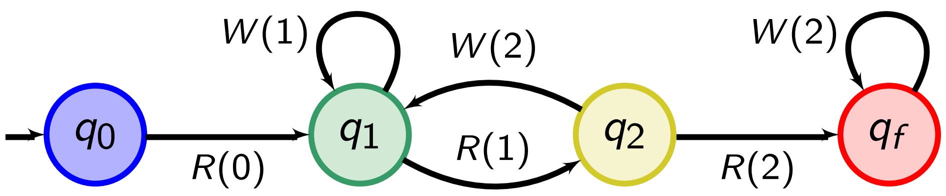

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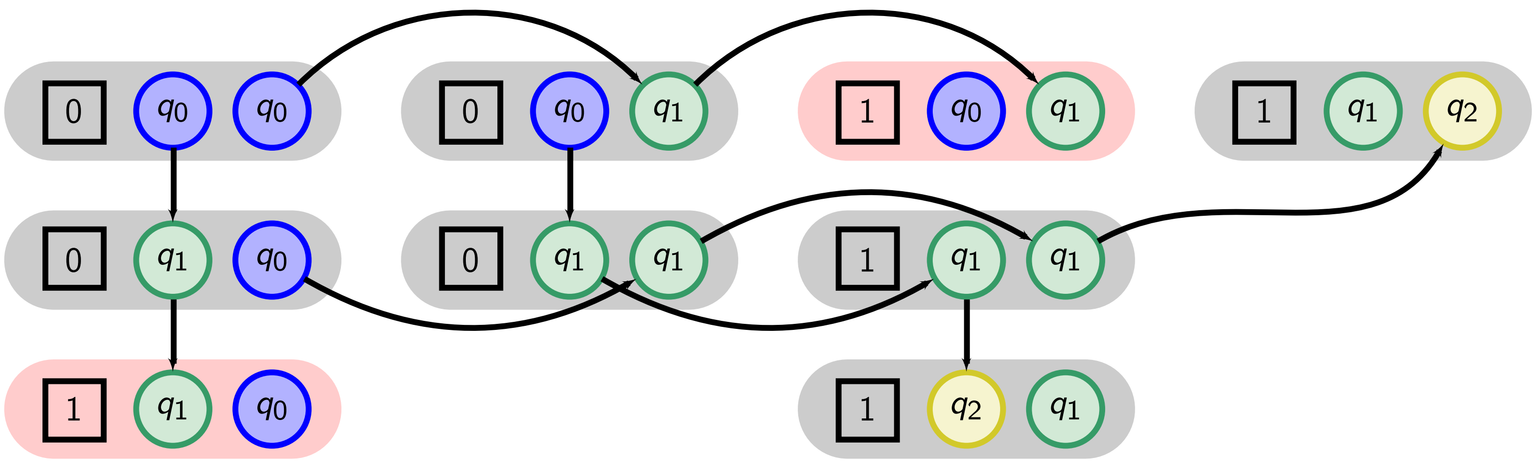

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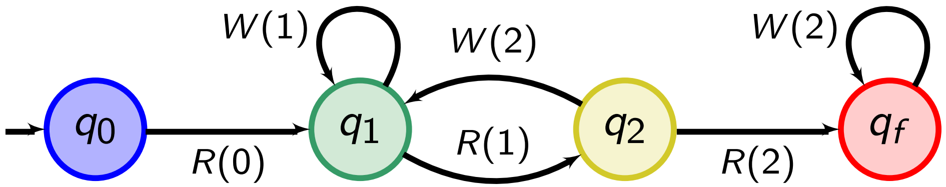

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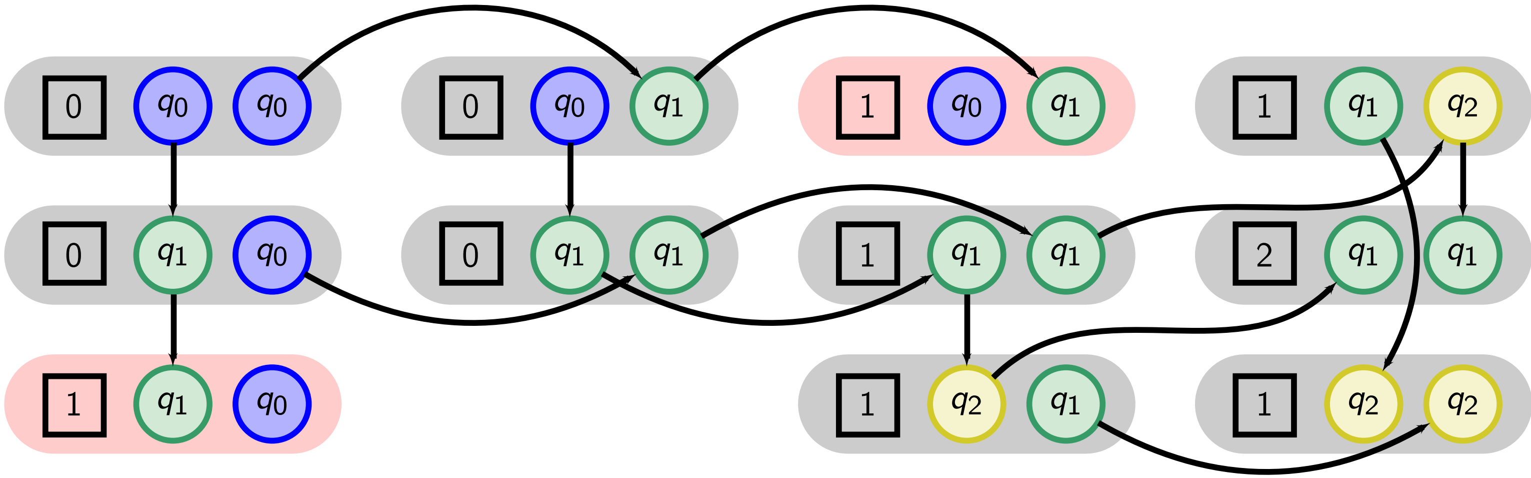

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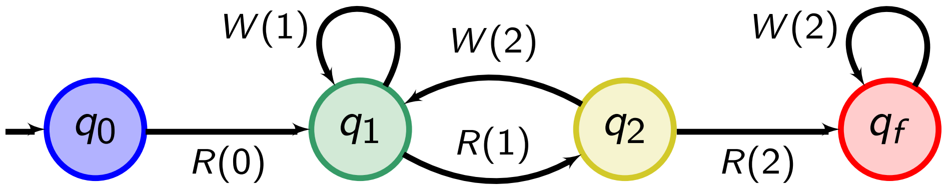

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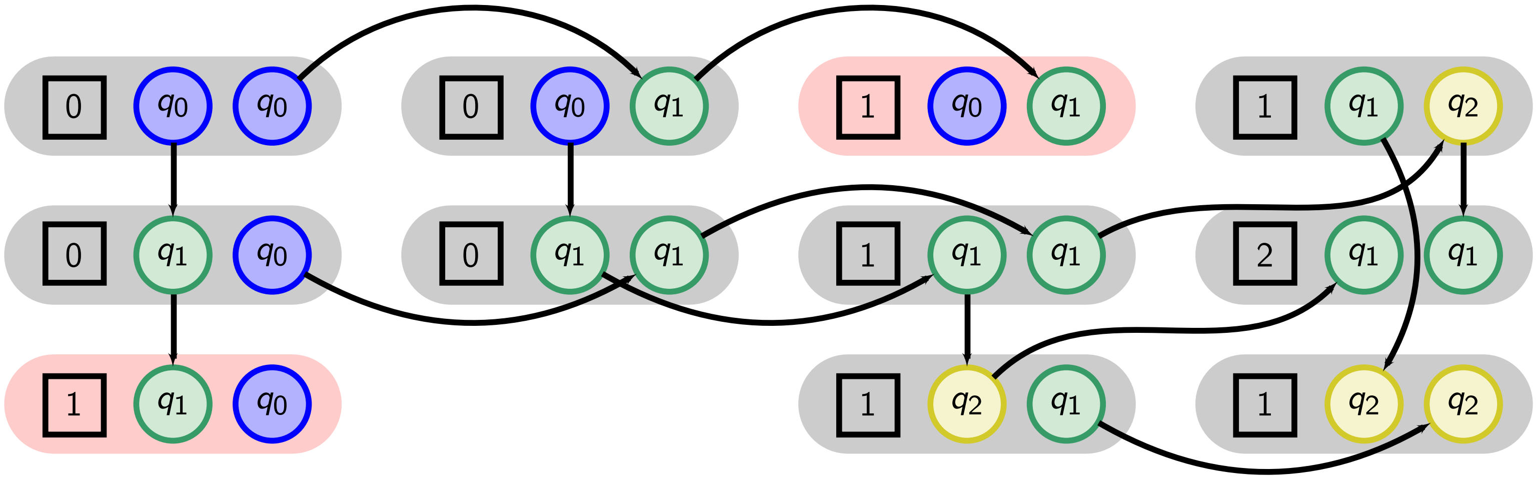

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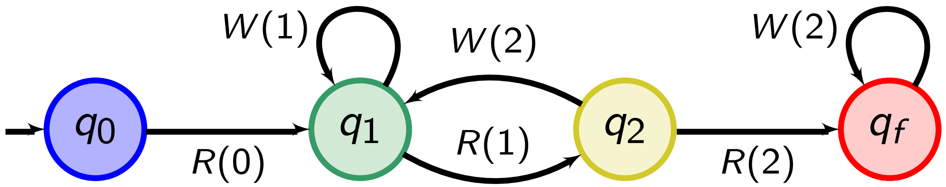

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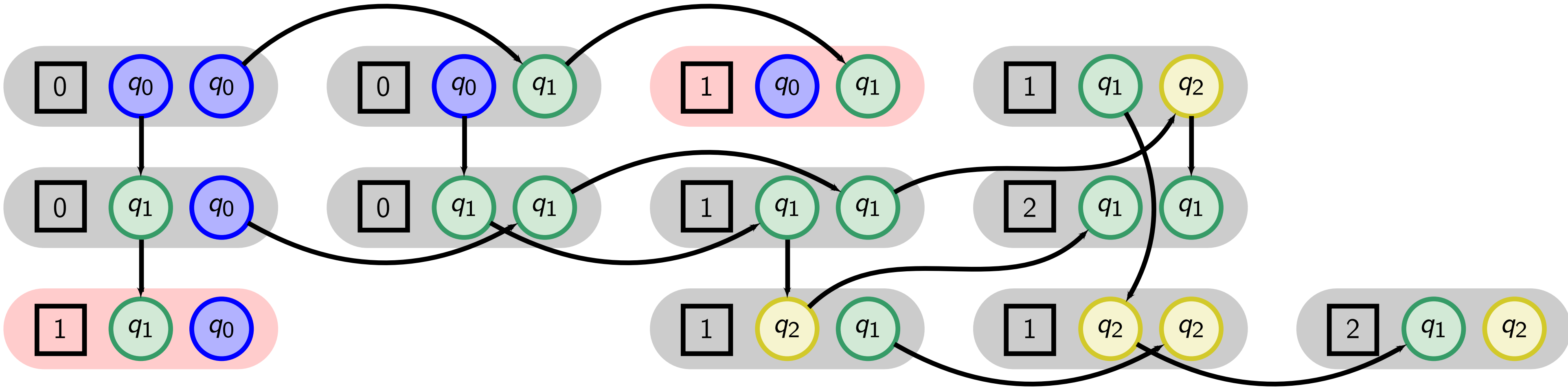

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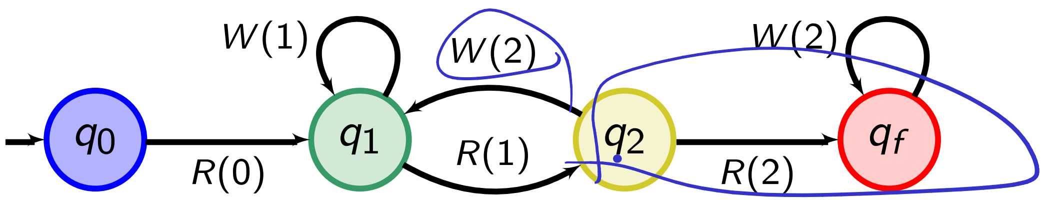

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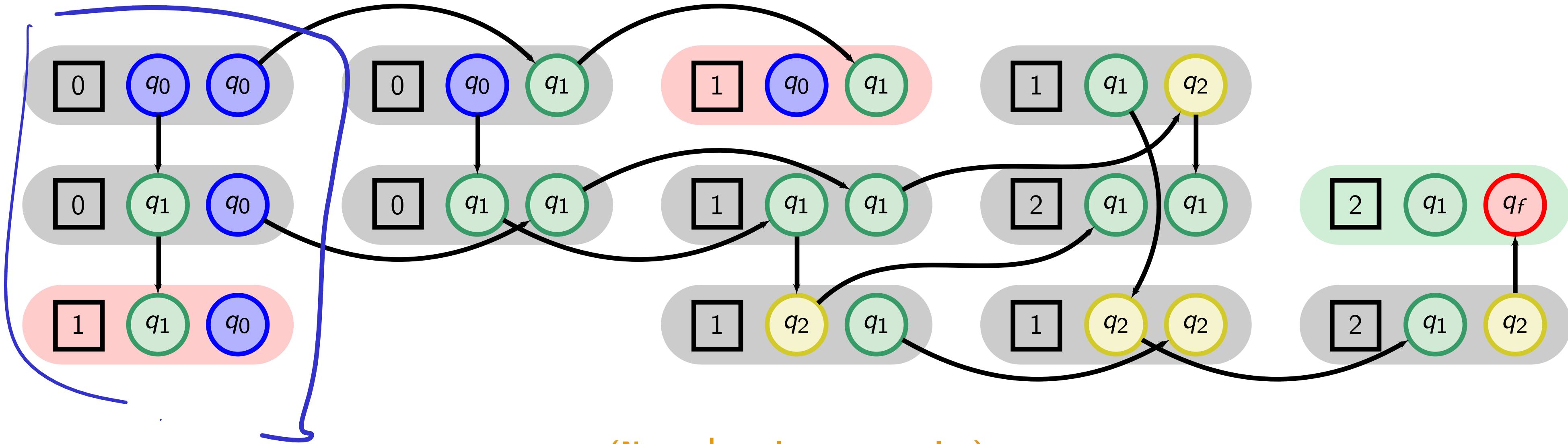

(Non-exhaustive construction)

- There exist paths from there, the processes in q_0 are trapped;
- There exist paths that reach q_f...
- ...and they require at least two processes.

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- How does the **scheduler** work ?

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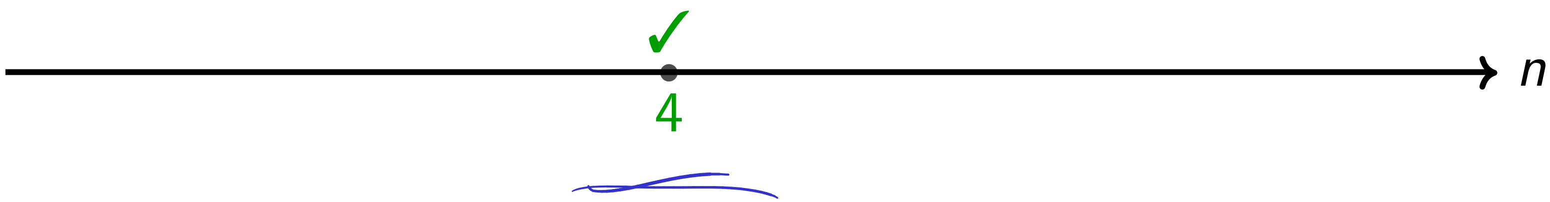

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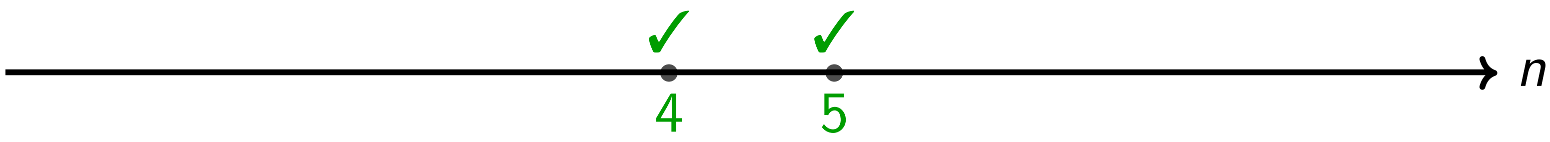

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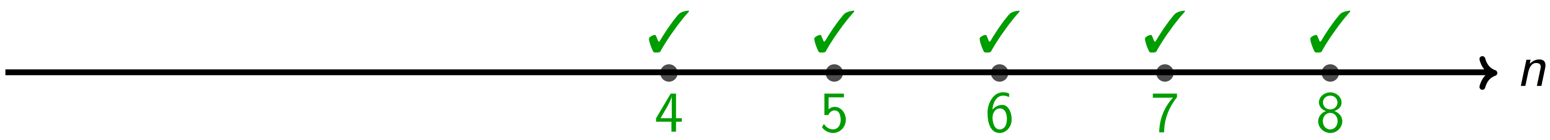

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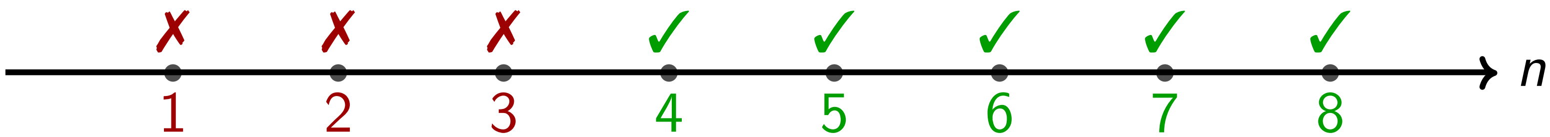

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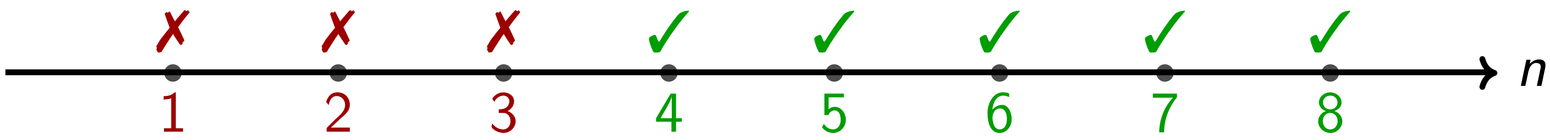

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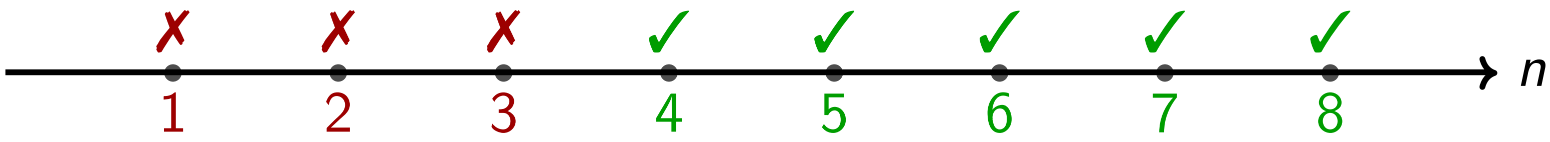

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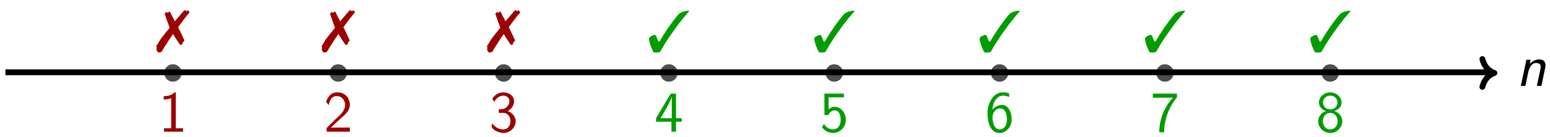

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Proof: check last appearance and dis-appearance of each control state...
Gives a bound on the runtime of the backward coverability algorithm.

Fair, Probabilistic scheduler

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- **Stochastic** behaviour (environment);
- Finite patterns cannot be repeated infinetely often;
- We consider **almost-sure** reachability: $\mathbb{P}_n(\diamond \uparrow q_f) \stackrel{?}{=} 1$

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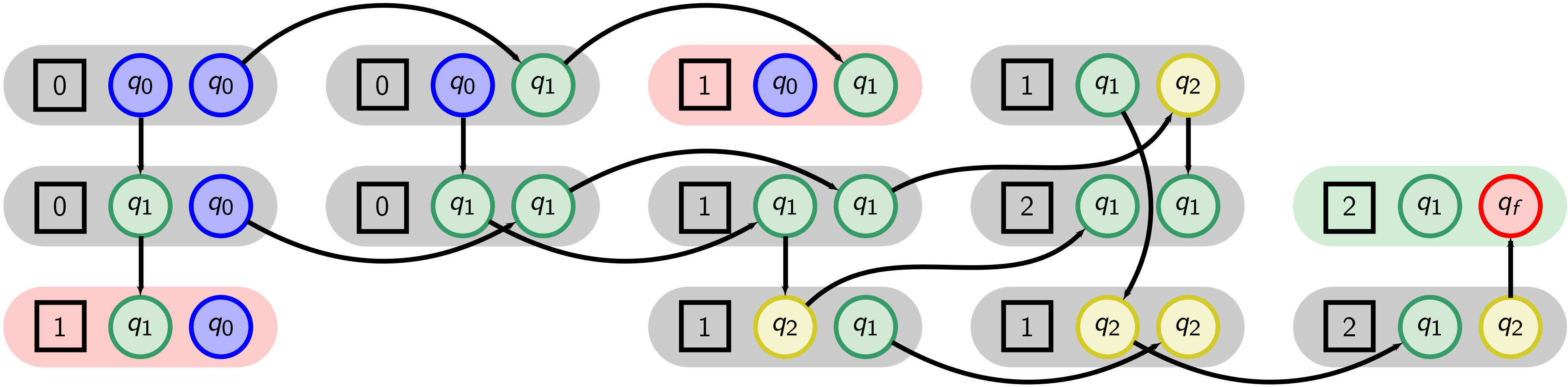

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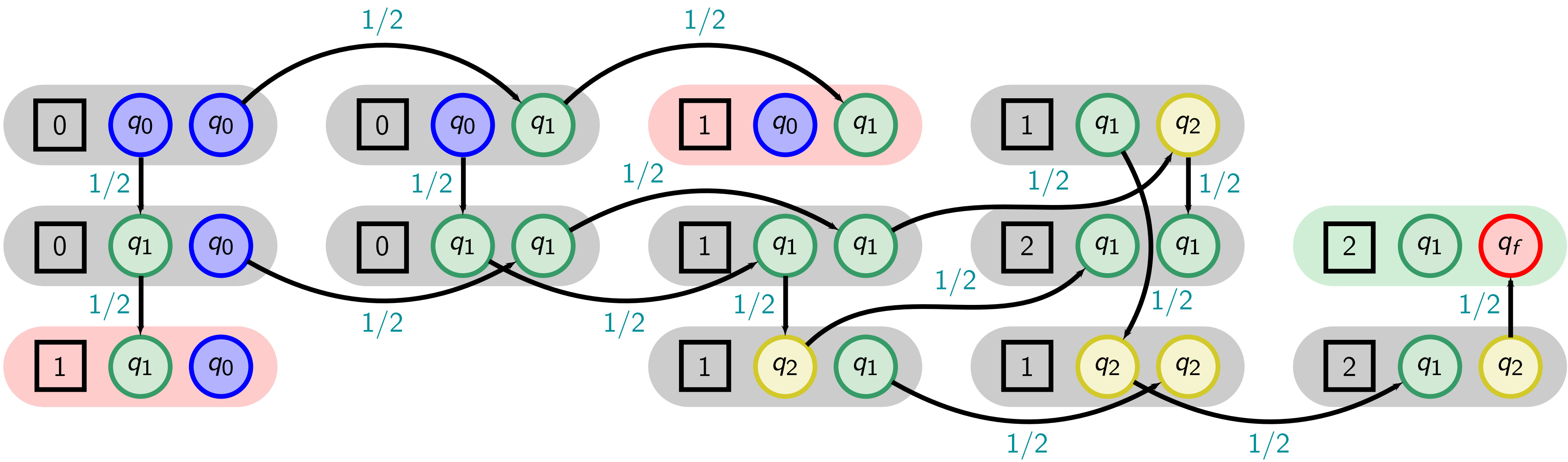

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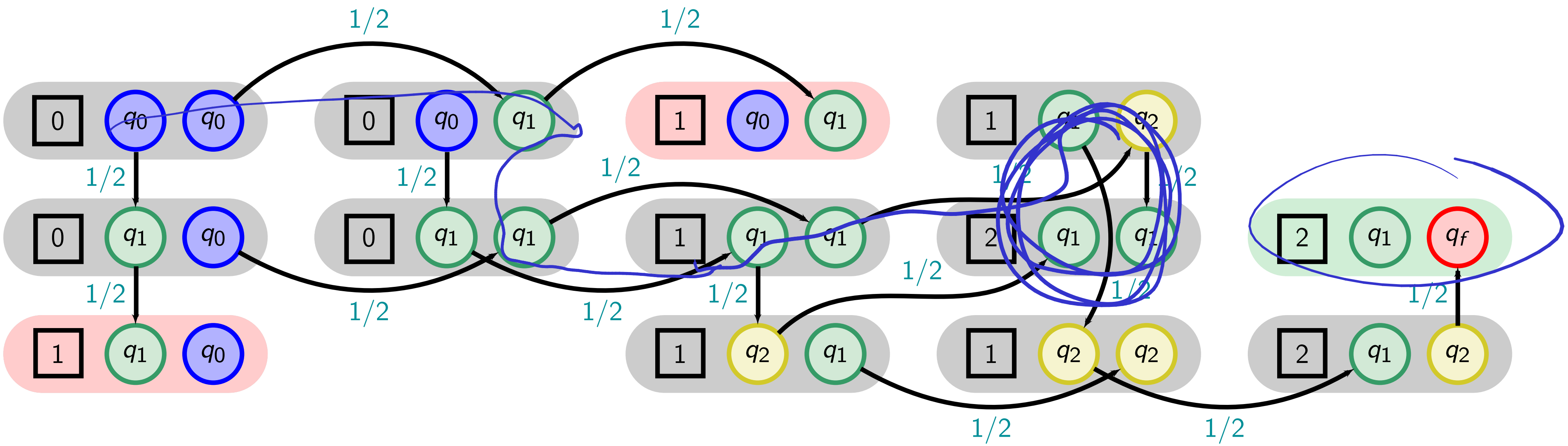
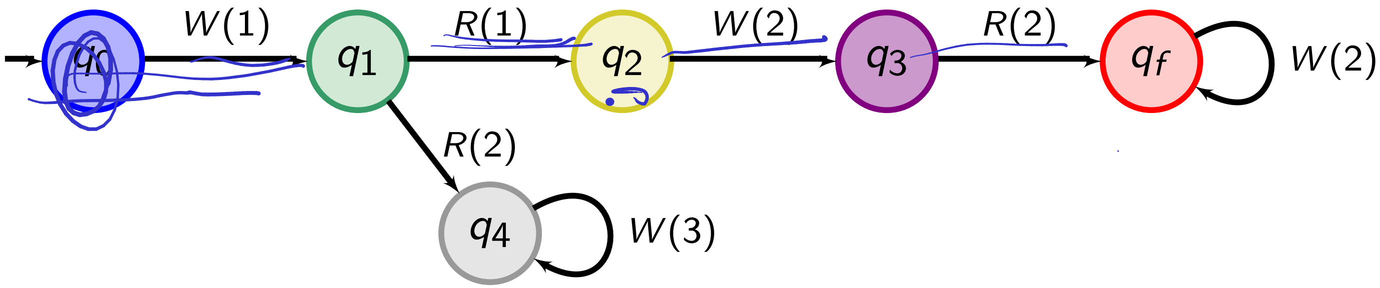


Qualitative property
$+$
Finite configuration space

$\Longrightarrow$

Exact probability values are **not** relevant.

Lack of monotonicity



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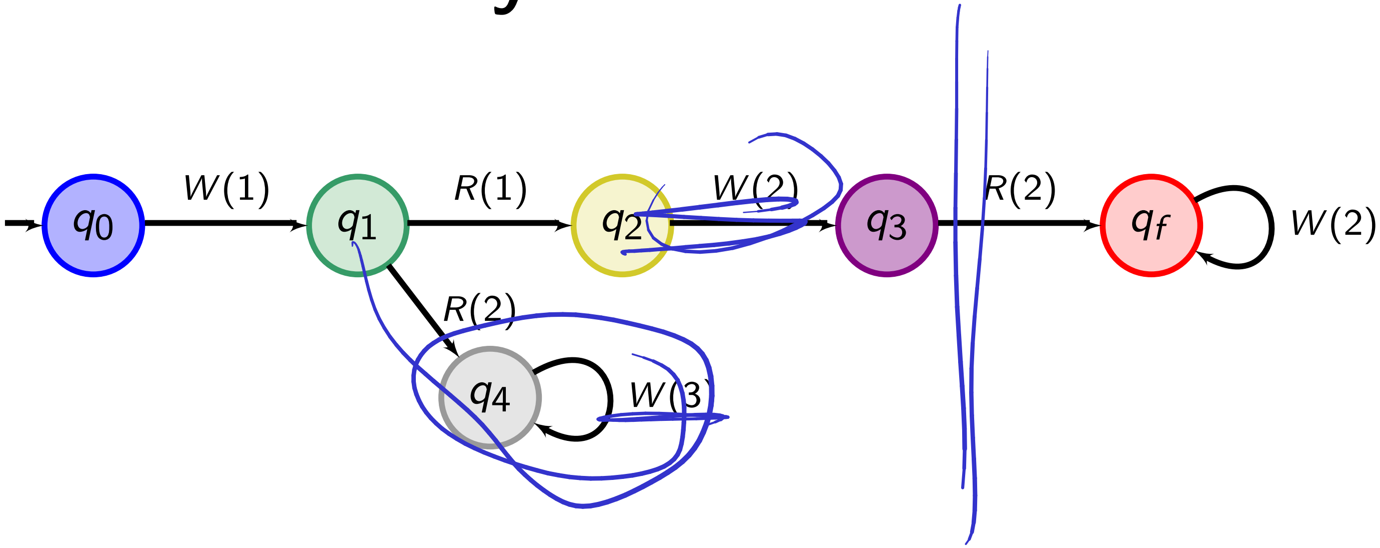

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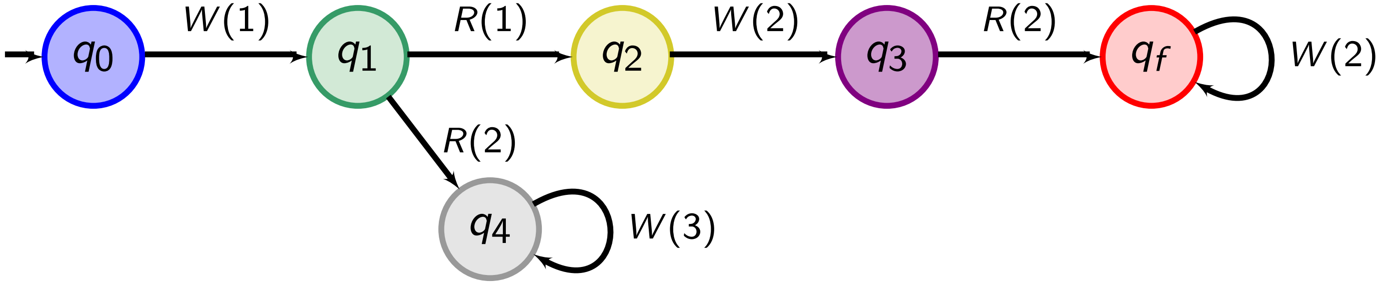

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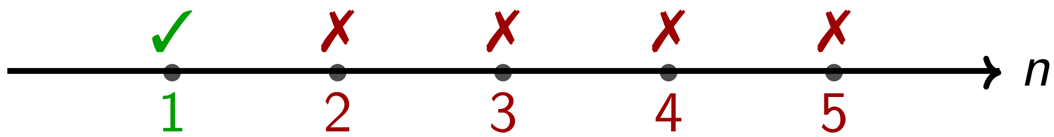

Probabilistic Cut-off and WQO relation

We study the cut-off for the following φ property:

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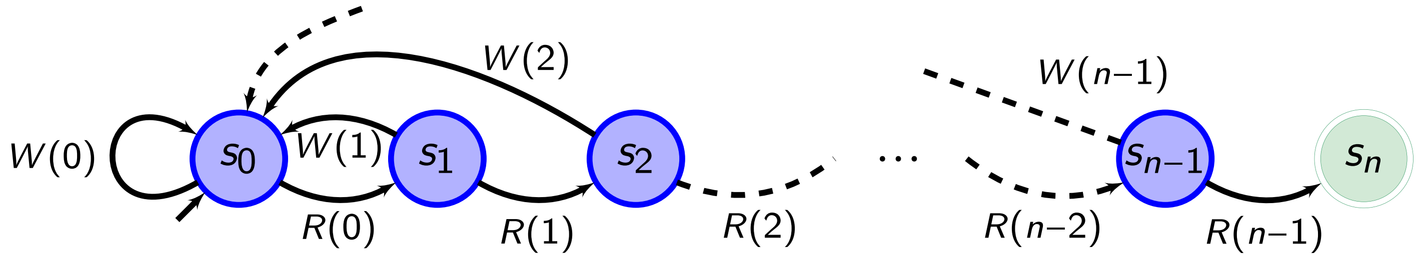
- For any reachable state s, there exists a path to reach $\uparrow q_f$ so the probability for this to happen is some positive number $f(s) > 0$.
- At any time point, the probability of eventually reaching $\uparrow q_f$ is at least $\max_{s \in S_k} f(s)$ which is **positive** since S_k is **finite**.

Examples



"Filter" protocol $\mathscr{F}_n$ for $n > 0$.

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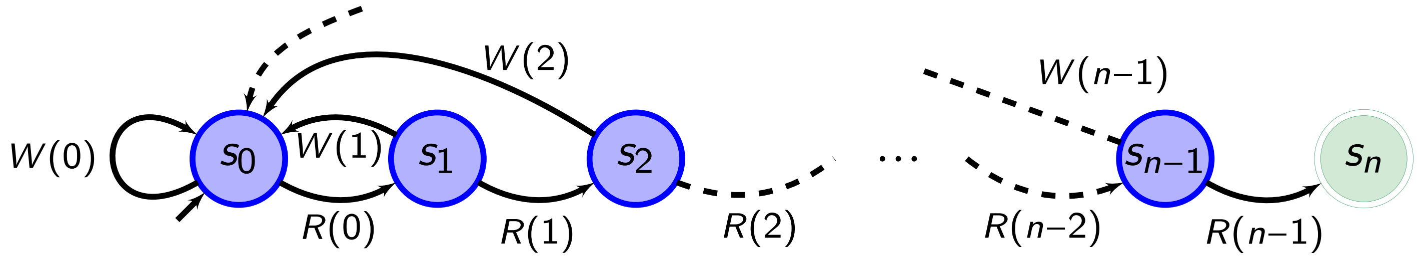

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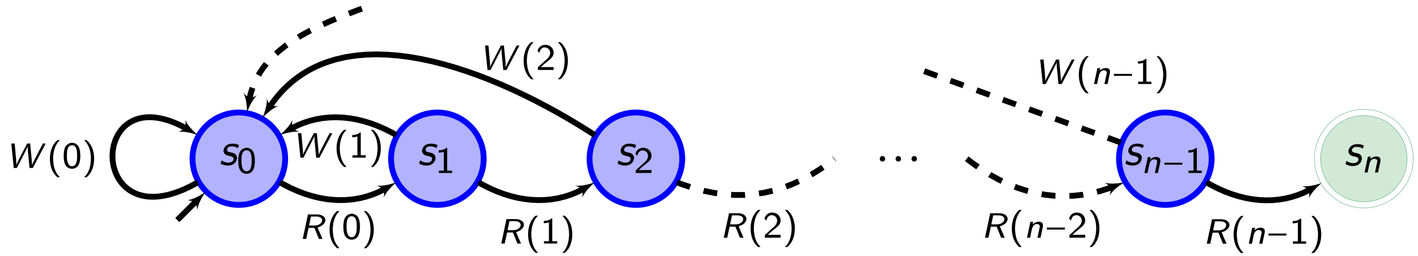

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$\Longrightarrow$ **Tight positive cut-off equal to n, i.e., linear in the protocol size.**

Existence of a cut-off

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The non-atomicity guarantees that when a process takes a transition, all processes in the same state can also take the same transition (with a non-zero probability).

$\Longrightarrow$ **a.k.a. copycat lemma.**

Existence: quick sketch (1/2)

- Write $\mathrm{Pre}^*(\uparrow q_f) \subseteq S$ the set of configurations that can reach q_f;
- Write $\mathrm{Post}*(I \cap S_{\geq n}) \subseteq S$ the set of reachable configurations of size $\geq n$;

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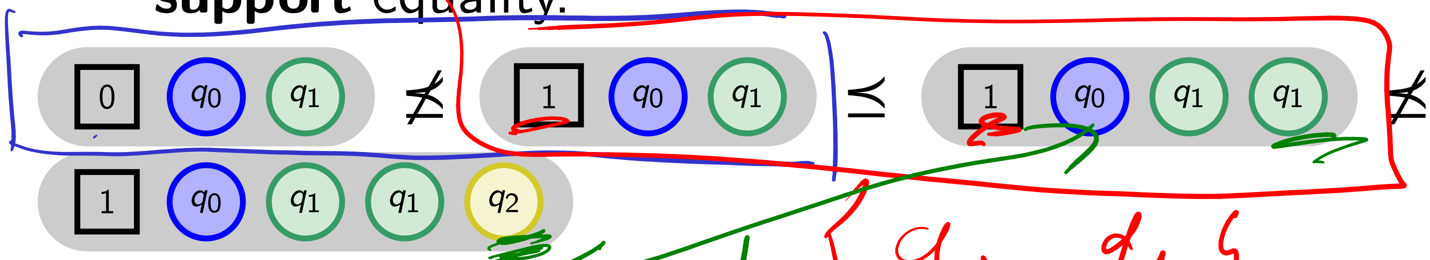

$\neq \{q_0, q_1\}$

$\preceq$ is still a WQO

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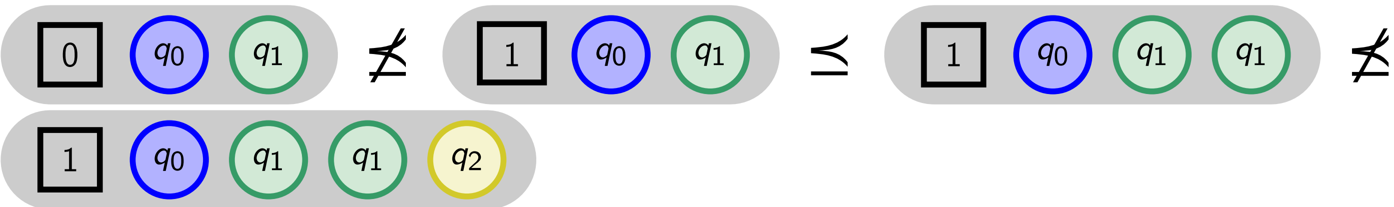

- $(S, \preceq)$ is a well-quasi-ordered set;
- Pre^* and $\mathrm{Post}^*(S_{\geq n})$ are upward-closed for any n;

Existence: quick sketch (2/2)

1. $\text{Post}^*(I \cap S_{\geq 1}) = \uparrow \{\theta_1, \ldots, \theta_l\}$ and $\text{Pre}^* = \uparrow \{\eta_1, \ldots, \eta_m\}$.

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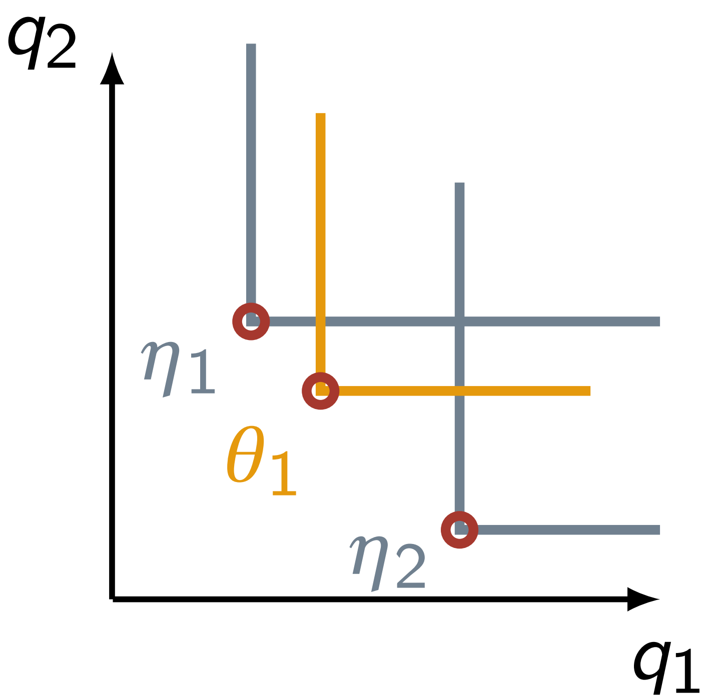

...intuitively, the goal is to check if elements of $\text{Post}^*(I \cap S_{\geq n})$ can enter Pre^* by adding sufficiently many processes in a given state.

Lemma 1. *Let $(S, I, \preceq, |.|)$ a parameterised system such that*

1. $(S, \preceq, \rightarrow^{-1})$ *is a WSTS;*

2. *For any $s, s' \in S$ with $s \preceq s'$ and for any $k \in [|s|, |s'|]$, there exists t of size k such that $s \preceq t \preceq s'$.*

Then for any upward closed set U, there exists N such that, either

$$\forall k \geq N, \mathrm{Post}^*(I \cap S_{=k}) \subseteq U$$

either,

$$\forall k \geq N, \mathrm{Post}^*(I \cap S_{=k}) \not\subseteq U$$

Proof. Let $K = \{k \mid \mathrm{Post}^*(I \cap S_{=k}) \not\subseteq U\}$.

- If K is finite, take $N = 1 + \max K$. Then $\forall k \geq N, \mathrm{Post}^*(I \cap S_{=k}) \subseteq U$.

- If K is infinite, for any $k \in K$, let $i_k \in I \cap S_{=k}$ and $s_k \in \mathrm{Post}^*(i_k) \setminus U$. Since $(S, \preceq)$ is a WQO, we can extract an infinite subset $K' \subset K$ such that for all $k_1, k_2 \in K'$ with $k_1 \leq k_2$, $i_{k_1} \preceq i_{k_2}$ but also (by extracting another subsequence), $x_{k_1} \preceq x_{k_2}$. We take $n = \min K$, for any $k \geq N$, since K' is infinite, there exists $k' \in K'$ such that $n \leq k \leq k'$. Since $x_n \preceq x_{k'}$ there exists x_k such that $x_n \preceq x_k \preceq x_{k'}$ with $|x_k| = k$. Since $i_n \rightarrow^* x_n \preceq x_k$ and $(S, \preceq, \rightarrow^{-1})$ is a WQO, there exists i_k such that $i_n \preceq i_k \rightarrow^* x_k$ hence $x_k \in \mathrm{Post}^*(I \cap S_{=k})$. Moreover, $\overline{U}$ is downward closed so $x_k \notin U$.

□